22 Triangles

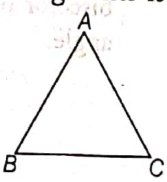
In this chapter, we will study about the congruency of triangle, criteria for congruency of triangles, similarity of triangles and some more proerties.

Triangle

A plane (closed) figure bounded by three line segments is called a triangle. It is denoted by Δ.

A ΔABC has

- three vertices, namely A, B and C.
- three sides, namely AB, BC and CA.
- three angles, namely $\angle A$, $\angle B$ and $\angle C$.

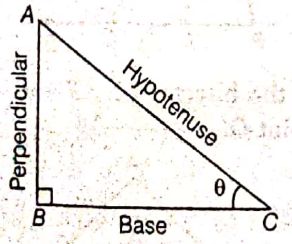


Classification of Triangle

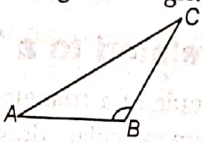
1. On the Basis of Angles

(i) **Right Angled Triangle** A triangle in which one of the angle measures 90° is called a right angled triangle. The side opposite to the right angle is called its hypotenuse and the remaining two sides are called as perpendicular and base depending upon conditions.

Here, ΔABC is a right angled triangle in which $\angle B = 90°$ and AC is hypotenuse.

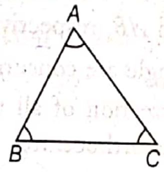


(ii) **Acute Angled Triangle** A triangle in which every the angles angle measure more than 0° but less than 90° is called an acute angled triangle.



Here, ΔABC is an acute angled triangle.

(iii) **Obtuse Angled Triangle** A triangle in which one of the angle measures more than 90° but less than 180° is called an obtuse angled triangle.

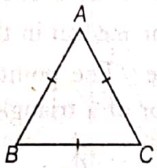


Here, ΔABC is an obtuse angled triangle and $\angle ABC$ is the obtuse angle.

2. On the Basis of Sides

(i) **Equilateral Triangle** A triangle having all sides equal is called an equilateral triangle. Here ΔABC, is an equilateral triangle in which $AB = BC = AC$.

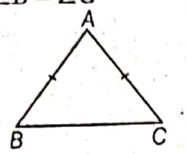
All angles are equal and are of measures 60°.



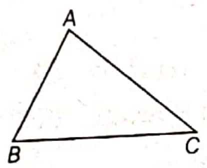
(*ii*) **Isosceles Triangle** A triangle in which two sides are equal, is called an isosceles triangle. Here, ΔABC is an isosceles triangle as $AB = AC$.

- Angle opposite to equal sides are equal.

i.e. $\angle B = \angle C$



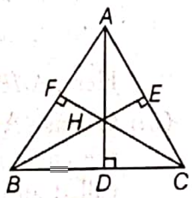
(*iii*) **Scalene Triangle** A triangle in which all the sides are of different lengths is called a scalene triangle. ΔABC is a scalene triangle as $AB \neq BC \neq AC$.



> The sum of the lengths of three sides of a triangle is called its perimeter.
> So, perimeter of $\Delta ABC = AB + BC + AC$

Some Term Related to a Triangle

Altitudes The altitude of a triangle is a line segment perpendicular drawn from vertex to the side opposite to it. The side on which the perpendicular is being drawn is called its base.

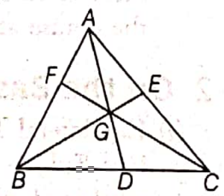


Here, AD, BE and FC are altitudes drawn on BC, AC and AB, respectively.

- Altitudes of a triangle are concurrent.
- The point of intersection of all the three altitudes of a triangle is called its orthocentre.

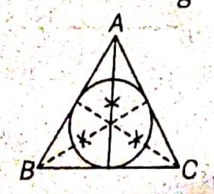
Medians A line segment joining the mid-point of that side with the opposite vertex.

Here, AD, BE and CF are medians.



- The medians of a triangle are concurrent.
- The point of intersection of all the three medians of a triangle is called its centroid. It is denoted by G.
- Centroid divides the median in the ratio 2 : 1.

Incentre of a Triangle The point of intersection of all the three angle bisector of a triangle is called its incentre.



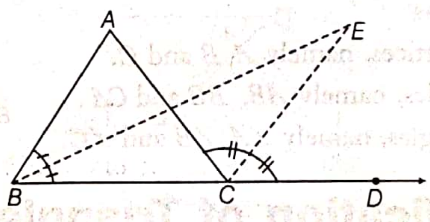
- The circle with centre I is called as incircle and radius is called as inradius denoted by 'r'.

Circumcentre of a Triangle The point of intersection of the perpendicular bisectors of the sides of a triangle is called its circumcentre.

Circle through it passing through A, B and C is called circumcircle. Radius of circumcircle is called circumradius denoted by R.

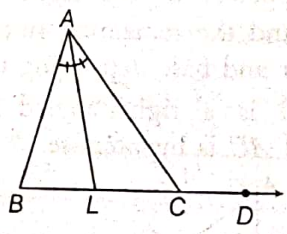
Some Important Results of a Triangle

- Angle opposite to two equal sides of an isosceles triangle are equal.
- If two angles of a triangle are equal, then the sides opposite to them are also equal.
- If two sides of a triangle are unequal, then the longer side has greater angle opposite to it.
- In a triangle, the greater angle has the longer side opposite to it.
- The sum of the angles of a triangle is 180°.
- If a side of a triangle is produced, then the exterior angle so formed is equal to the sum of the two interior opposite angles.
- An exterior angle of a triangle is greater than either of the interior opposite angles.
- The internal bisector of one base angle and the external bisector of the other is equal to one half of the vertical angle.

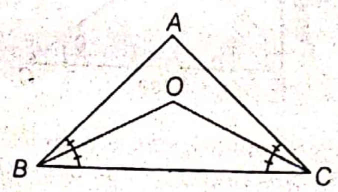


Here, $\angle E = \frac{1}{2}\angle A$

- The side BC of ΔABC is produced to D. The bisector of $\angle A$ meets BC in L. Then, $\angle ABC + \angle ACD = 2\angle ALC$.

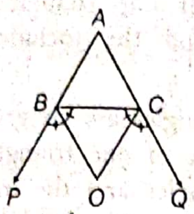


- In a ΔABC the bisector of $\angle B$ and $\angle C$ intersect each other at a point O.



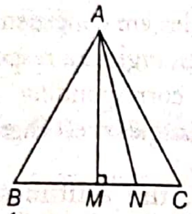
$\angle BOC = 90° + \frac{1}{2}\angle A$

- In a ΔABC, the side AB and AC are produced to P and Q, respectively. The bisectors of $\angle PBC$ and $\angle QCB$ intersect at a point O.



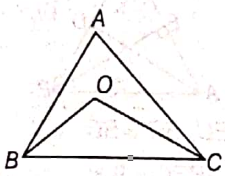
Then, $\angle BOC = 90° - \frac{1}{2}\angle A$

- In ΔABC, $\angle B > \angle C$. If AN is the bisector of $\angle BAC$ and $AM \perp BC$, then



$$\angle MAN = \frac{1}{2}\angle B - \angle C$$

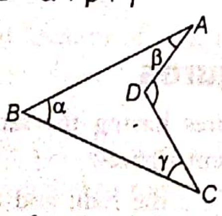
- The bisectors of the base angles of a triangle can never enclose a right angle.
- A triangle is an isosceles if and only if any two altitudes are equal.



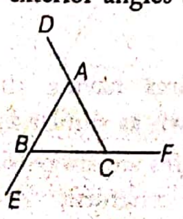
Here, OB and OC are the bisectors of $\angle B$ and $\angle C$.

But $\angle BOC \neq 90°$.

- In figure, $\angle ADC = \alpha + \beta + \gamma$

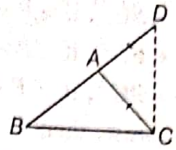


- If the three sides of a triangle be produced in order, then the sum of all the exterior angles so formed is 360°.



So, $\angle DAB + \angle EBC + \angle ACF = 360°$

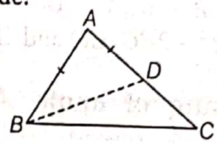
- The sum of any two sides of a triangle is greater than its third side.



Here, in ΔABC

$$AB + AC > BC$$
$$AB + BC > AC$$
$$BC + AC > AB$$

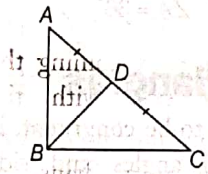
- The difference between any two sides of a triangles is less than its third side.



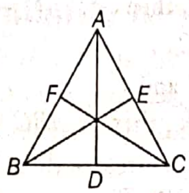
Here, in ΔABC

$$AC - AB < BC$$
$$BC - AC < AB$$
$$BC - AB < AC$$

- If the bisector of the vertical angle of a triangle bisects the base, then that triangle is an isosceles.
- If the altitude from one vertex of a triangle bisects the opposite side, then the triangle is an isosceles.
- The perpendiculars drawn from the vertices of equal angles of an isosceles triangle to the opposite sides are equal.
- Median of equilateral triangle are equal.
- If D is the mid-point of the hypotenuse AC of a right angled ΔABC. Then, $BD = \frac{1}{2}AC$



- The sum of any two sides of a triangle is greater than twice the median drawn to the third side.
- Perimeter of a triangle is greater than the sum of its three medians.



So, $AB + BC + AC > AD + BE + CF$.

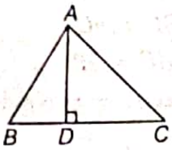
- In a ΔABC, $\angle B = 90°$, an obtuse angle or an acute angle according it

$$AC^2 = AB^2 + BC^2, \text{ if } \angle B = 90°$$
$$AC^2 > AB^2 + BC^2, \text{ if } \angle B > 90°$$
$$AC^2 < AB^2 + BC^2, \text{ if } \angle B < 90°$$

- In ΔABC, if $\angle B$ is obtuse, then

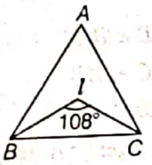
$$AC^2 = AB^2 + BC^2 + 2BC \cdot AD$$

- In ΔABC, if $\angle B$ is acute, then



$$AB^2 = BC^2 + AC^2 - 2BC \cdot CD \text{ and } AD \perp BC$$

Ex. 1. The measure of angle A in the figure given below is



(a) 54° (b) 18°
(c) 36° (d) None of these

Sol. (c) Here, I is the incentre of the ΔABC.

$\therefore$ BI and CI are the bisectors of $\angle B$ and $\angle C$, then

we know that, $\angle BIC = 90° + \frac{1}{2}\angle A$

$\Rightarrow$ $108° = 90° + \frac{1}{2}\angle A$

$\Rightarrow$ $\frac{1}{2}\angle A = 108° - 90° = 18°$

$\angle A = 36°$

Congruent Triangles

Two triangles are said to be congruent, if both are exactly of same size i.e. all angles and sides are equal to corresponding angles and sides of other.

- *Every triangle is congruent to itself*
  $\Delta ABC \cong \Delta ABC$
- *If* $\Delta ABC \cong \Delta DEF$, *then*
  $\Delta DEF \cong \Delta ABC$
- *If* $\Delta ABC \cong \Delta DEF$ *and*
  $\Delta DEF \cong \Delta PQR$, *then* $\Delta ABC \cong \Delta PQR$

ı congruent triangles, corresponding parts are equal and e write in short 'CPCT' i.e. corresponding part of ngruent triangles.

Criteria for Congruence of Triangles

Theorem 1 Two triangles are congruent, if two sides and the included angle If one triangle are equal to the corresonding sides and the included angle of other triangle, (SAS)

Theorem 2 Two triangles are congruent. If two angles and the included side of one triangle are equal to the corresponding two angles and the included side of the other triangles. (ASA)

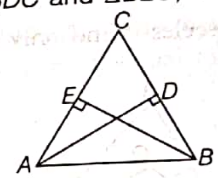
Theorem 3 Two triangles are congruent. If three sides of one triangle are respectively equal to the three sides of the other triangle. Then, two triangles are congruent. (SSS)

Theorem 4 Two triangles are congruent. If the hypotenuse and other side of one triangle are respectively equal to the hypotenuse and the corresponding side of the other triangle. Then, two triangles are congruent. (RHS)

Ex. 2. In a ΔABC, the altitudes BD and CE are equal and $\angle A = 36°$. What is the value of the $\angle B$?

(a) 72° (b) 84°
(c) 18° (d) 36°

Sol. (a) For the ΔBDC and ΔBEC,



$BD = EC$, $BC = BC$

and $\angle BEC = \angle BDC = 90°$

Thus, $\Delta BEC \cong \Delta BDC$ [by SAS rule]

$\therefore$ $\angle B = \angle C = \frac{180° - 36°}{2} = 72°$ each

Congruent Figures

The geometrical figures having the same shape and size are known as congruent figures.

e.g. Two circles of the same radii and two squares of the same sides are congruent.

Similar Figures

The geometrical figures having the same shape but different sizes are known as similar figures.

- The congruent figures are always similar but two similar figures need not be congruent.

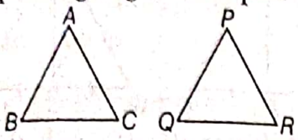
  e.g. Any two circles are similar. Any two rectangles are similar.

Similar Triangles

Two triangles are said to be similar to each other, if

(*i*) their corresponding sides are proportional.

(*ii*) their corresponding angles are equal.



Here, ΔABC and ΔPQR are similar triangles.

$\therefore \quad \angle A = \angle P, \ \angle B = \angle Q, \ \angle C = \angle R$

and $\quad \dfrac{AB}{PQ} = \dfrac{BC}{QR} = \dfrac{AC}{PR}$

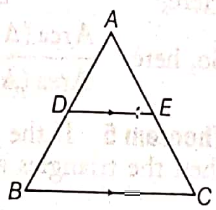
Then $\Delta ABC \sim \Delta PQR$

where symbol ~ is read as, 'is similar to'.

Some Results on Similar Triangles

Theorem 1 If a line is drawn parallel to one side of a triangle to intersect the other two sides in distinct point, then it divides these sides in the same ratio. It is also called thales theorem.

Here, $DE || BC$, then

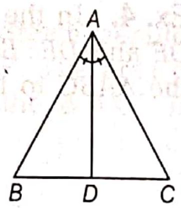


$$\frac{AD}{DB} = \frac{AE}{EC}$$

or $$\frac{AD}{AB} = \frac{AE}{AC}$$

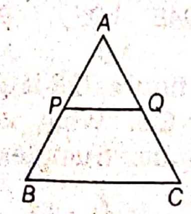
or $$\frac{AB}{BD} = \frac{AC}{EC}$$

Theorem 2 The internal bisector of an angle of a triangle divides the opposite sides internally in the ratio of the sides containing the angle.



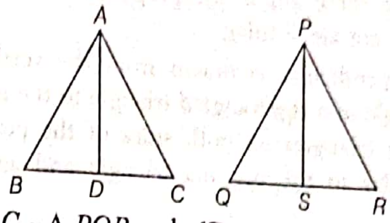
Here, AD is internal bisector of $\angle A$, then $\dfrac{AB}{AC} = \dfrac{BD}{DC}$

Theorem 3 The line joining the mid-points of any two sides of a triangle is parallel to the third side and is half of the third side.



Here, P and Q are mid-point of AB and AC. So, $PQ = \dfrac{1}{2}BC$.

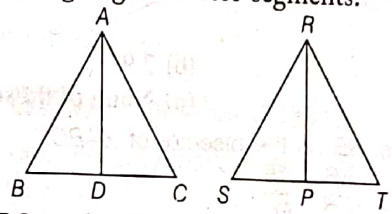
Theorem 4 If two triangles are equiangular, then the ratio of their corresponding sides is the same as the ratio of the corresponding altitudes.



Here, $\Delta ABC \sim \Delta PQR$ and AD and PS are altitude on BC and QR, respectively,

then $$\frac{BC}{QR} = \frac{AD}{PS}$$

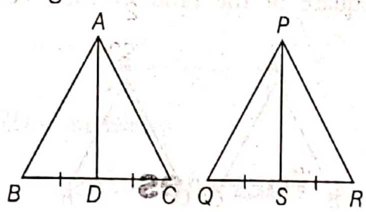
Theorem 5 If two triangles are equiangular, then the ratio of the corresponding sides is the same as the ratio of the corresponding angle bisector segments.



Here, ΔABC and ΔRST are equiangular/similar and AD, RP are the angle bisectors of $\angle A$ and $\angle R$.

Then, $$\frac{BC}{ST} = \frac{AD}{RP}$$

Theorem 6 If two triangles are equiangular, then the ratio of the corresponding sides is the same as the ratio of the corresponding medians.



Here, ΔABC and ΔPQR are equiangular and AD, PS are the medians, then

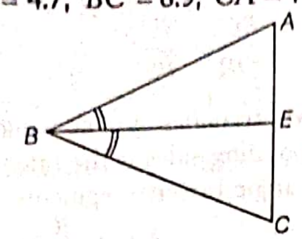
$$\frac{BC}{QR} = \frac{AD}{PS}$$

Criteria for Similarity of Two Triangles

1. **AAA similarity** If two angles of one triangle are respectively equal to the corresponding two angles of another triangle, then the two triangles are similar.
2. **SSS similarity** If the corresponding sides of two triangles are proportional, then their corresponding angles are equal and hence the two triangles are similar.

3. **SAS similarity** If one angle of a triangle is equal to the corresponding angle of the orther triangle and the sides including these angle one proportional, then the two triangles are are similar.
4. If a perpendicular is drawn from the verthex of the right angle of a right angled triangle to the hypotenuse, then the triangles on both sides of the perpendicular are similar to the original triangle and also to each other.

Ex. 3. If $AB = 4.7$, $BC = 8.9$, $CA = 11.5$, then EA is



(a) 6.07 (b) 3.97
(c) 2.37 (d) None of these

Sol. *(b)* Since, BE is the bisector of $\angle ABC$.

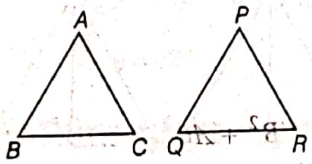
$$\therefore \quad \frac{AB}{BC} = \frac{AE}{EC}$$

$$\Rightarrow \quad \frac{4.7}{8.9} = \frac{AE}{11.5 - AE} \quad [\because EC = AC - AE]$$

$$\Rightarrow \quad AE = 3.97$$

Areas of Similar Triangles

Theorem 1 The ratio of areas of two similar triangles is equal to the square of the ratio of their corresponding sides.

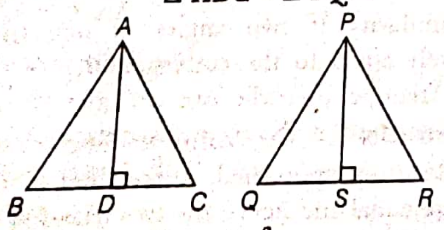


Here, $\Delta ABC \sim \Delta PQR$

$$\frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta PQR)} = \frac{AB^2}{PQ^2} = \frac{BC^2}{QR^2} = \frac{AC^2}{PR^2}$$

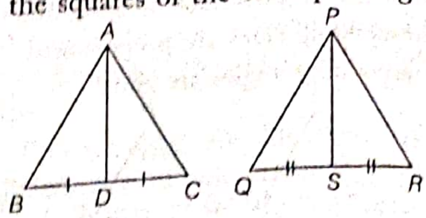
Theorem 2 The areas of two similar triangles is equal to the ratio of the squares of corresponding altitudes.

Here, $\Delta ABC \sim \Delta PQR$



Then, $$\frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta PQR)} = \frac{AD^2}{PS^2}$$

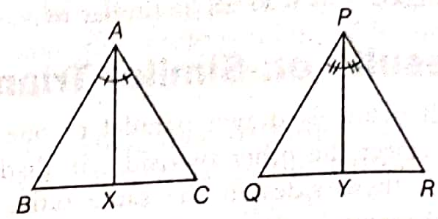
Theorem 3 The areas of two similar triangles is equal to the ratio of the squares of the corresponding medians.



Here, $\Delta ABC \sim \Delta PQR$,

then $$\frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta PQR)} = \frac{AD^2}{PS^2}$$

Theorem 4 The areas of two similar triangles is equal to the ratio of squares of the corresponding angle bisector segments.



Here, $\Delta ABC \sim \Delta PQR$

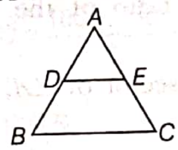
So, here $$\frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta PQR)} = \frac{AX^2}{PY^2}$$

Theorem 5 If the areas of two similar triangles are equal, then the triangles are congruent.

or

Equal and similar triangles are congruent.

Ex. 4. In the figure given below, BC is parallel to DE and $DE : BC = 3 : 5$. What is the ratio of area of the ΔABC to that of ΔDEA?

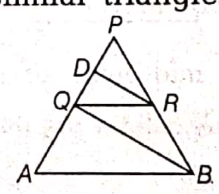


(a) 3 : 1 (b) 5 : 3 (c) 9 : 2 (d) 25 : 9

Sol. *(d)* $\because$ Given, $DE : BC = 3 : 5$

$$\therefore \quad \frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DAE} = \left(\frac{BC}{DE}\right)^2 = \frac{25}{9}$$

Ex. 5. In a given figure, QR is parallel to AB and DR is parallel to QB. What is the number of distinct pairs of similar triangles?



(a) 1 (b) 2 (c) 3 (d) 4

Sol. (c) Since, QR is parallel to AB.

$\therefore \quad \Delta PQR \sim \Delta QPB$

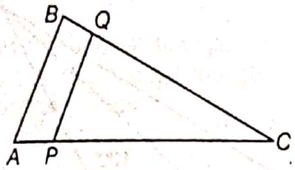
Also, DR is parallel to QB.

$\Delta PQB \sim \Delta QDR$

Again, $DR \parallel QB$ and $QR \parallel AB$

$\therefore \quad \Delta DQR \sim \Delta AQB$

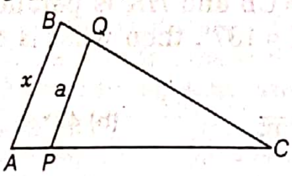
Ex. 6. In the given triangle, AB is parallel to PQ. $AP = c$, $PC = b$, $PQ = a$, $AB = x$. What is the value of x?



(a) $a + \frac{ab}{c}$ (b) $a + \frac{bc}{a}$

(c) $b + \frac{ca}{b}$ (d) $a + \frac{ac}{b}$

Sol. (d) In ΔABC and ΔPQC.



$$\therefore \quad \frac{PC}{AC} = \frac{PQ}{AB} \Rightarrow \frac{b}{c+a} = \frac{a}{x}$$

$$\therefore \quad x = \frac{a(c+b)}{b} = \frac{ac}{b} + a$$

Ex. 7. In a triangle, a line XY is drawn parallel to BC meeting AB in X and AC in Y. The area of the ΔABC is 2 times the area of the ΔAXY. In what ratio X divides AB?

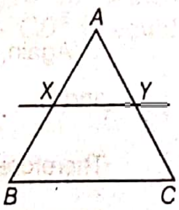
(a) $1 : \sqrt{2}$

(b) $\sqrt{2} : 1$

(c) $(\sqrt{2} - 1) : 1$

(d) $1 : (\sqrt{2} - 1)$

Sol. (d) $\because \frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta AXY)} = \frac{AB^2}{AX^2}$



$$\Rightarrow \frac{2\,\text{Area}(\Delta AXY)}{\text{Area}(\Delta AXY)} = \frac{AB^2}{AX^2}$$

$$\Rightarrow \frac{2}{1} = \frac{AB^2}{AX^2}, \frac{AB}{AX} = \sqrt{2}$$

$$\Rightarrow AB = \sqrt{2}\,AX$$

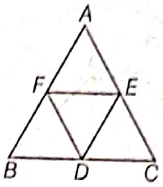
$$\Rightarrow AX + BX = \sqrt{2}\,AX$$

$$\Rightarrow BX = AX(\sqrt{2} - 1)$$

$$\therefore \frac{AX}{BX} = \frac{1}{\sqrt{2} - 1}$$

So, X divides AB in $1 : (\sqrt{2} - 1)$.

Theorem 7 The line segment joining the mid-points of the sides of a triangle form four triangles, each of which is similar to the original triangle.



Here, D, E and F are mid-point of BC, AC and AB. Then, here ΔAFE, ΔFBD, ΔEDC and ΔDEF is similar to ΔABC.

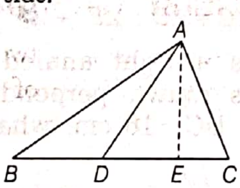
Here, also

$$\frac{\text{Area}(\Delta DEF)}{\text{Area}(\Delta ABC)} = \frac{DE^2}{AB^2} = \frac{\left(\frac{1}{2}AB\right)^2}{AB^2} = \frac{1}{4}$$

So, area (ΔDEF) : area $(\Delta ABC) = 1 : 4$

Some Other Useful Facts

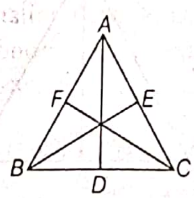
- The area of the equilateral triangle described on the side of a square is half the area of the equilateral triangle described on its diagonal.
- In any triangle, the sum of the square of any two sides is equal to twice the square of half of the third side together with twice the square of the median which bisect the third side.



Here, AD is median, so

$$AB^2 + AC^2 = 2AD^2 + 2\left(\frac{1}{2}BC\right)^2$$

- In a ΔABC, three times the sum of the squares of the sides of a triangle is equal to four times the sum of the squares of the medians of the triangle.



So, in triangle, if AD, BE, FC are the medians, then

$$3(AB^2 + BC^2 + AC^2) = 4(AD^2 + BE^2 + CF^2)$$

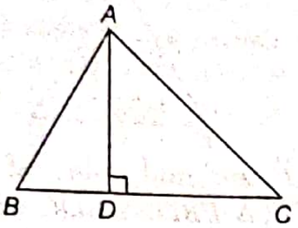
- In an equilateral ΔABC, the side BC is trisected at D. Then,

$$9\,AD^2 = 7\,AC^2$$

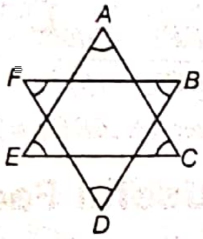
Pythagoras Theorem

In a right angled triangle, the square of the hypotenuse is equal to the sum of the squares on the other two sides.

i.e., $AB^2 + BC^2 = AC^2$



Ex. 8. In the diagram given below what is the sum of all the angles $\angle A$, $\angle B$, $\angle C$, $\angle D$, $\angle E$ and $\angle F$?



(a) 120° (b) 180°
(c) 290° (d) 360°

Sol. **(d)** Since, sum of the angles of a triangle is 180°.

In ΔAEC, $\angle A + \angle C + \angle E = 180°$ ...(i)

and In ΔBDF, $\angle B + \angle D + \angle F = 180°$ ...(ii)

On adding the Eqs. (i) and (ii), we get

$\angle A + \angle B + \angle C + \angle D + \angle E + \angle F = 180° + 180° = 360°$

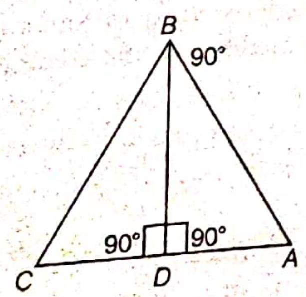
Ex. 9. ABC is a right angled triangle, where $\angle B = 90°$. BD is drawn perpendicular to AC. If $AD = 9$ cm and $DC = 16$ cm, what is the measure of AB?

(a) 15 cm (b) 18 cm
(c) 16 cm (d) 9.5 cm

Sol. **(a)** given, in right angled ΔABC

$AD = 9$ cm, $DC = 16$ cm

$\therefore BD^2 = AD \times DC = 9 \times 16 = 144$



$\therefore BD = 12$

Now, $AB^2 = BD^2 + AD^2 = 144 + 81 = 225$

$\therefore AB = 15$ cm

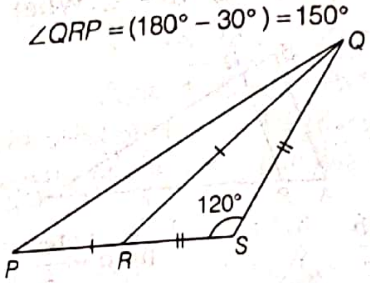
Ex. 10. In a ΔPQS, R is a point on PS such that $PR = QR$ and $QS = RS$. If $\angle RSQ = 120°$, what is the measure of $\angle QPR$?

(a) 30° (b) 15°
(c) 45° (d) None of these

Sol. **(b)** $\because RS = SQ$

$\therefore \angle QRS = \frac{1}{2}(180° - 120°) = 30°$

$\therefore \angle QRP = (180° - 30°) = 150°$

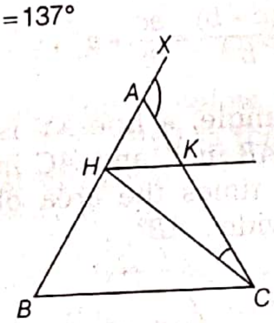


Hence, $\angle QPR = \frac{1}{2}(180° - 150°) = 15°$ [$\because PR = RQ$]

Ex. 11. ABC is an isosceles triangle in which $AB = AC$, $CH = CB$ and HK is parallel to BC. If the exterior $\angle CAX = 137°$, then what is the measure of $\angle HCK$?

(a) $68\frac{1}{2}°$ (b) 43°
(c) $25\frac{1}{2}°$ (d) 137°

Sol. **(c)** $\because \angle CAX = 137°$



$\therefore \angle ABC = \frac{1}{2}(137°)$

$= 68\frac{1}{2}°$

$\therefore$ Again, $BC = CH$

and $\angle ABC = 68\frac{1}{2}°$

Therefore, $\angle CHB = 68\frac{1}{2}°$

Therefore, $\angle HCB = 43°$

Hence, $\angle HCX = 68\frac{1}{2}° - 43°$

$= 25\frac{1}{2}°$