CHAPTER 23

Indefinite Integrals

Integration

The process of finding the function whose derivative is known is called *integration*. Integration is the inverse of differentiation and is therefore also called anti-differentiation.

Indefinite Integrals

If $F(x)$ is a differentiable function of x and

$$\frac{d}{dx}[F(x)] = f(x) \quad \text{...(i)}$$

then, we write

$$\int f(x)\,dx = F(x)$$

The function $F(x)$ is called the integral or the anti-derivative of $f(x)$ with respect to x. The function $f(x)$ is called the integrand and x is called the variables of integration of the integral. The symbol $\int$ is called the symbol of integration.

Since, the derivative of a constant is zero, we have

$$\frac{d}{dx}[F(x)+C] = \frac{d}{dx}F(x) + \frac{d}{dx}(C)$$
$$= f(x) \quad \text{...(ii)}$$

Hence, we, obtain

$$\int f(x)\,dx = F(x) + C \quad \text{...(iii)}$$

where C is any arbitrary constant and is called the constant of integration. By assuming different values of C we obtain different values of the integral. *i.e.*, the integral of a function $f(x)$ as given by Eq. (iii) has infinite number of values (one for each value of C) and is therefore called the indefinite integral of $f(x)$ with respect to x.

Integration of Some Elementary Functions

(i) $\frac{d}{dx}(x) = 1;\ \int dx = x + C$

(ii) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n;\ \int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$

(iii) $\frac{d}{dx}(\log|x|) = \frac{1}{x};\ \int \frac{1}{x}dx = \log|x| + C$

(iv) $\frac{d}{dx}(e^x) = e^x;\ \int e^x dx = e^x + C$

(v) $\frac{d}{dx}\left(\frac{a^x}{\log a}\right) = a^x;\ \int a^x dx = \frac{a^x}{\log a} + C$

(vi) $\frac{d}{dx}(\cos x) = -\sin x;\ \int \sin x\,dx = -\cos x + C$

(vii) $\frac{d}{dx}(\sin x) = \cos x;\ \int \cos x\,dx = \sin x + C$

(viii) $\frac{d}{dx}(\sec x) = \sec x \tan x;\ \int \sec x \tan x\,dx = \sec x + C$

(ix) $\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x;$

$\int \text{cosec}\, x \cot x\,dx = -\text{cosec}\, x + C$

(x) $\frac{d}{dx}(\tan x) = \sec^2 x;\ \int \sec^2 x\,dx = \tan x + C$

(xi) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x;\ \int \text{cosec}^2 x\,dx = -\cot x + C$

(xii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}};\ \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$

(xiii) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}};\ \int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$

(xiv) $\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$; $\int\frac{dx}{1+x^2}=\tan^{-1}x+C$

(xv) $\frac{d}{dx}(\cot^{-1}x)=-\frac{1}{1+x^2}$; $\int\frac{dx}{1+x^2}=-\cot^{-1}x+C$

(xvi) $\frac{d}{dx}(\sec^{-1}x)=\frac{1}{|x|\sqrt{x^2-1}}$; $\int\frac{dx}{|x|\sqrt{x^2-1}}=\sec^{-1}x+C$

(xvii) $\frac{d}{dx}(\text{cosec}^{-1}x)=\frac{-1}{|x|\sqrt{x^2-1}}$;

$$\int\frac{dx}{|x|\sqrt{x^2-1}}=-\text{cosec}^{-1}x+C$$

Properties of Integration

(i) $\int\alpha f(x)dx=\alpha\int f(x)dx$, where α is any arbitrary constant.

(ii) $\int[f(x)\pm g(x)]dx=\int f(x)\,dx\pm\int g(x)\,dx$

(iii) $\int f'(g(x))g'(x)dx=\int(g(x))$

Example 1. Evaluate $\int\frac{x^{1/3}-x^{1/2}+1}{\sqrt{x}}dx$

(a) $\frac{5}{6}x^{5/6}-x+2x^{1/2}+C$

(b) $\frac{6}{5}x^{5/6}-x+2x^{1/2}+C$

(c) $\frac{6}{5}x^{5/6}-x+\frac{1}{2}x^{1/2}+C$

(d) None of the above

Sol. (b) We have,

$$\int\frac{x^{1/3}-x^{1/2}+1}{x^{1/2}}dx=\int\frac{x^{1/3}}{x^{1/2}}dx-\int\frac{x^{1/2}}{x^{1/2}}dx+\int\frac{1}{x^{1/2}}dx$$
$$=\int x^{-1/6}dx-\int dx+\int x^{-1/2}dx$$
$$=\frac{x^{5/6}}{(5/6)}-x+\frac{x^{1/2}}{(1/2)}+C$$
$$=\frac{6}{5}x^{5/6}-x+2x^{1/2}+C$$

Example 2. Evaluate $\sqrt{1+\sin 2x}\,dx$

(a) $\sin x+C$
(b) $\cos x+C$
(c) $\sin x-\cos x+C$
(d) $\sin x+\cos x+C$

Sol. (c) $\sqrt{(1+\sin 2x)}dx$

$$=\int\sqrt{(\cos^2x+\sin^2x+2\sin x\cos x)}\,dx$$
$$=\int\sqrt{(\cos x+\sin x)^2}dx=\int\cos x\,dx+\int\sin x\,dx$$
$$=\sin x-\cos x+C$$

Integration by Substitution

Sometimes, a given integral can be reduced to a standard form by using a suitable substitution. Consider the integral

$$I=\int f(g(x))\cdot g'(x)\,dx$$

Here we substitute $g(x)=t$ and $g'(x)\,dx=dt$

Then, given integral reduces to

$$I=\int f(t)\,dt$$

In standard form and we can easily simplify it.

Some Important Substitution

	Expression	Substitution
(i)	a^2+x^2	$x=a\tan\theta$ or $a\cot\theta$
(ii)	a^2-x^2	$x=a\sin\theta$ or $a\cos\theta$
(iii)	x^2-a^2	$x=a\sec\theta$ or $a\,\text{cosec}\,\theta$
(iv)	$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x=a\cos 2\theta$

Example 3. What is the value of the integral

$$\int e^{\left(\frac{x^2+1}{x}\right)}dx-\int\frac{e^{\left(\frac{x^2+1}{x}\right)}}{x^2}dx?$$ *(NDA 2005 II)*

(a) e^x+C
(b) $e^{\left(\frac{x^2+1}{x}\right)}+C$
(c) $xe^{\left(\frac{x^2+1}{x}\right)}+C$
(d) $\left(x+\frac{1}{x}\right)e^x+C$

Sol. (b) Let $I=\int e^{\left(\frac{x^2+1}{x}\right)}dx-\int\frac{e^{\left(\frac{x^2+1}{x}\right)}}{x^2}dx$

$$=\int\left(1-\frac{1}{x^2}\right)e^{\left(x+\frac{1}{x}\right)}dx$$

Put $x+\frac{1}{x}=t$

$\Rightarrow$ $\left(1-\frac{1}{x^2}\right)dx=dt$

$\therefore$ $I=\int e^t\,dt=e^t+C=e^{\left(x+\frac{1}{x}\right)}+C=e^{\left(\frac{x^2+1}{x}\right)}+C$

Example 4. What is the value of $\int\frac{e^x(1+x)}{\sin^2(xe^x)}dx$? *(NDA 2007 II)*

(a) $-e^x\cot x+C$
(b) $\cos^2(xe^x)+C$
(c) $\log\sin(xe^x)+C$
(d) $-\cot(xe^x)+C$

Sol. (d) Let $I=\int\frac{e^x(1+x)}{\sin^2(xe^x)}dx$

Put $xe^x = t \Rightarrow e^x(1+x)dx = dt$

$\therefore$
$$I = \int \frac{dt}{\sin^2 t} = \int \text{cosec}^2 t\, dt$$
$$= -\cot t + C$$
$$= -\cot(xe^x) + C$$

Example 5. What is the value of $\int \frac{\sin x}{\sqrt{\sin^2 x - \sin^2 \alpha}} dx$?

(NDA 2005 II)

(a) $\sin^{-1}(\sec \alpha \cos x) + C$ (b) $\cos^{-1}(\sec \alpha \cos x) + C$
(c) $\sinh^{-1}(\sec \alpha \cos x) + C$ (d) $\cosh^{-1}(\sec \alpha \cos x) + C$

Sol. (b) Let
$$I = \int \frac{\sin x}{\sqrt{\sin^2 x - \sin^2 \alpha}} dx$$
$$= \int \frac{\sin x}{\sqrt{1 - \cos^2 x - \sin^2 \alpha}} dx$$
$$= \int \frac{\sin x}{\sqrt{\cos^2 \alpha - \cos^2 x}} dx$$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt$
$$I = -\int \frac{1}{\sqrt{\cos^2 \alpha - t^2}} dt$$
$$= \cos^{-1}\left(\frac{t}{\cos \alpha}\right) + C$$
$$= \cos^{-1}(\sec \alpha \cos x) + C$$

Example 6. Evaluate the following integrals

(I) $\cot^m x\, \text{cosec}^2 x\, dx, m \neq -1$

(a) $\log|\cot x| + C$ (b) $-\log|\cot x| + C$
(c) $\frac{-1}{m+1} \cot^{m+1} x + C$ (d) None of these

(II) $\int \frac{e^{\tan^{-1} x}}{1+x^2} dx$

(a) $e^{\tan^{-1} x} + C$ (b) $e^{\tan^{-1} x} + x + C$
(c) $e^{\tan x} + \frac{1}{x} + C$ (d) None of these

(III) $\int \frac{\sin 2x\, dx}{p\cos^2 x + q\sin^2 x}$

(a) $(p-q)\log|p + (q-p)\sin^2 x| + C$
(b) $\frac{1}{q-p}\log|p + (q-p)\sin^2 x| + C$
(c) $\frac{1}{p-q}\log|p - (q-p)\sin^2 x| + C$
(d) None of the above

Sol. (I) (c) Let $I = \int \cot^m x\, \text{cosec}^2 x\, dx$

Put $\cot x = y \Rightarrow (-\text{cosec}^2 x)dx = dy$

$\therefore$
$$I = -\int y^m dy$$
$$= \frac{-1}{(m+1)} y^{m+1} + C$$
$$= \frac{-1}{(m+1)} \cot^{m+1} x + C, \text{ if } m \neq -1$$

(II) (a) Substituting $\tan^{-1} x = y$. We get $\frac{dx}{1+x^2} = dy$.

$\therefore$
$$I = \int e^y\, dy = e^y + C = e^{\tan^{-1} x} + C$$

(III) (b) $I = \int \frac{\sin 2x}{p\cos^2 x + q\sin^2 x} dx = \int \frac{2\sin x \cos x}{p + (q-p)\sin^2 x} dx$

Put $\sin^2 x = y \Rightarrow 2\sin x \cos x\, dx = dy$

$\therefore$
$$I = \int \frac{dy}{p + (q-p)y}$$
$$= \frac{1}{q-p}\log|p + (q-p)y| + C$$
$$= \frac{1}{q-p}\log|p + (q-p)\sin^2 x| + C$$

Integrals Involving Trigonometric Functions

(i) $\int \tan x\, dx = \log|\sec x| + C$

(ii) $\int \cot x\, dx = \log|\sin x| + C$

(iii) $\int \sec x\, dx = \log|\sec x + \tan x| + C$
$$= \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + C$$

(iv) $\int \text{cosec}\, x\, dx = \log|\text{cosec}\, x - \cot x| + C$

Example 7. Evaluate the integral $\int \frac{1 - \sin x}{1 - \cos x} dx$.

(a) $\cot \frac{x}{2} - 2\log|\sin \frac{x}{2}| + C$
(b) $\cot \frac{x}{2} + 2\log|\sin \frac{x}{2}| + C$
(c) $-\cot \frac{x}{2} + 2\log|\sin \frac{x}{2}| + C$
(d) None of the above

Sol. (d)
$$I = \int \frac{1 - 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{1 - \left(1 - 2\sin^2\left(\frac{x}{2}\right)\right)} dx$$
$$= \int \frac{1 - 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{2\sin^2\left(\frac{x}{2}\right)} dx$$

$$=\frac{1}{2}\int \operatorname{cosec}^2\left(\frac{x}{2}\right)dx-\int \cot\left(\frac{x}{2}\right)dx$$

$$=-\cot\left(\frac{x}{2}\right)-2\log\left|\sin\left(\frac{x}{2}\right)\right|+C$$

Example 8. Evaluate the integral $\int \frac{dx}{x\cos^2(1+\log x)}$.

(a) $\log|1+\tan x|+C$ (b) $\log|1-\tan x|+C$
(c) $\tan(1+\log x)+C$ (d) $\tan(1-\log x)+C$

Sol. (c) Substituting $1+\log x=y$, we get $\frac{dx}{x}=dy$.

$$\therefore \quad I=\int \frac{dy}{\cos^2 y}=\int \sec^2 y\,dy$$

$$=\tan y+C=\tan(1+\log x)+C$$

Integrals of the Form $\int \sin^p x \cos^q x\, dx$

To evaluate the integrals of the form $\int \sin^p x\cos^q x\,dx$, we use the following procedure.

(i) If p is an odd integer and q is any real number substitute $\cos x=y$.

(ii) If q is an odd integer and p is any real number, substitute $\sin x=y$.

(iii) If p and q are both even integers, then expression $\sin^p x$ and $\cos^q x$ in terms of cosines of multiple angles.

Example 9. Evaluate the following integrals

(I) $\int \sin^3 x\sqrt{\cos x}\,dx$ (II) $\int \sin^2 x\cos^3 x\,dx$,

(a) $\frac{2}{3}\cos^{3/2}x+\frac{2}{7}\cos^{7/2}x+C$

(b) $-\frac{2}{3}\cos^{3/2}x+\frac{2}{7}\cos^{7/2}x+C$

(c) $\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+C$

(d) $\frac{1}{3}\sin^3 x+\frac{1}{5}\sin^5 x+C$

Sol. (I) (b) $I=\int \sin^3 x\sqrt{\cos x}\,dx=\int(\sin x)(\sin^2 x)\sqrt{\cos x}\,dx$

$$=\int \sin x(1-\cos^2 x)\sqrt{\cos x}\,dx$$

$$=-\int(1-y^2)\sqrt{y}\,dy$$

(Put $\cos x=y \Rightarrow -\sin x\,dx=dy$)

$$=-\int y^{1/2}dy+\int y^{5/2}dy$$

$$=-\frac{2}{3}y^{3/2}+\frac{2}{7}y^{7/2}+C$$

$$=-\frac{2}{3}(\cos x)^{3/2}+\frac{2}{7}(\cos x)^{7/2}+C$$

(II)(c) $I=\int \sin^2 x\cos^3 x\,dx=\int \sin^2 x(\cos^2 x)\cos x\,dx$

$$=\int \sin^2 x(1-\sin^2 x)\cos x\,dx$$

$$=\int y^2(1-y^2)dy$$

(Put $\sin x=y \Rightarrow \cos x\,dx=dy$)

$$=\frac{1}{3}y^3-\frac{1}{5}y^5+C$$

$$=\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+C$$

Integrals of Some Particular Functions

(i) $\int \frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

(ii) $\int \frac{dx}{a^2-x^2}=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+C$

(iii) $\int \frac{dx}{x^2-a^2}=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C$

(iv) $\int \frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\left(\frac{x}{a}\right)+C$ or $-\cos^{-1}\left(\frac{x}{a}\right)+C$

(v) $\int \frac{dx}{\sqrt{x^2-a^2}}=\log|x+\sqrt{x^2-a^2}|+C$

(vi) $\int \frac{dx}{\sqrt{x^2+a^2}}=\log|x+\sqrt{x^2+a^2}|+C$

(vii) $\int \sqrt{a^2-x^2}\,dx=\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+C$

(viii) $\int \frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)+C$ or $-\frac{1}{a}\operatorname{cosec}^{-1}\left(\frac{x}{a}\right)+C$

Example 10. Evaluate $\int \frac{x+1}{\sqrt{9-4x^2}}dx$

(a) $\frac{1}{4}\sqrt{9-4x^2}+\frac{1}{2}\sin^{-1}\left(\frac{2}{3}x\right)+C$

(b) $\frac{1}{4}\sqrt{9-4x^2}-\frac{1}{2}\sin^{-1}\left(\frac{2}{3}x\right)+C$

(c) $-\frac{1}{4}\sqrt{9-4x^2}+\frac{1}{2}\sin^{-1}\left(\frac{2}{3}x\right)+C$

(d) $-\frac{1}{4}\sqrt{9-4x^2}-\frac{1}{2}\sin^{-1}\left(\frac{2}{3}x\right)+C$

Sol. (c) Let $I=\int \frac{x\,dx}{\sqrt{9-4x^2}}+\int \frac{dx}{\sqrt{9-4x^2}}=I_1+I_2$ (say)

Now, $I_1=\int \frac{x\,dx}{\sqrt{9-4x^2}}$

Put $9-4x^2=y^2 \Rightarrow x\,dx=-\left(\frac{1}{4}\right)y\,dy$

$$I_1=-\frac{1}{4}\int\frac{y\,dy}{\sqrt{y^2}}=-\frac{1}{4}\int dy$$

$$=-\frac{1}{4}y+C_1=-\frac{1}{4}\sqrt{9-4x^2}+C_1$$

$$I_2=\int\frac{dx}{\sqrt{9-4x^2}}=\frac{1}{2}\sin^{-1}\left(\frac{2}{3}x\right)+C_2$$

$\therefore$ $$I=-\frac{1}{4}\sqrt{9-4x^2}+\frac{1}{2}\sin^{-1}\left(\frac{2}{3}x\right)+C$$

Where $C=C_1+C_2$ is another arbitrary constant.

Example 11. Evaluate the integral

$$\int\frac{dx}{x^4+a^2}.$$

(a) $\frac{1}{2a}\left[\tan^{-1}\frac{x^2-a}{x\sqrt{2a}}+\frac{1}{2}\log\frac{x^2-\sqrt{2a}x+a}{x^2+\sqrt{2a}\,x+a}\right]+C$

(b) $\frac{1}{2a\sqrt{2a}}\left[\tan^{-1}\frac{x^2-a}{x\sqrt{2a}}-\frac{1}{2}\log\frac{x^2-\sqrt{2a}x+a}{x^2+\sqrt{2a}\,x+a}\right]+C$

(c) $\frac{1}{2\sqrt{2a}}\left[\tan^{-1}\frac{x^2-a}{x\sqrt{2a}}-\frac{1}{2}\log\frac{x^2+\sqrt{2a}x+a}{x^2-\sqrt{2a}\,x-a}\right]+c$

(d) $\frac{1}{\sqrt{2a}}\left[\tan^{-1}\frac{x^2-a}{x\sqrt{2a}}+\frac{1}{2}\log\frac{x^2-\sqrt{2a}x+a}{x^2+\sqrt{2a}\,x+a}\right]+c$

Sol. (b) Let $I=\int\frac{dx}{x^4+a^2}=\frac{1}{2a}\int\frac{(x^2+a)-(x^2-a)}{x^4+a^2}dx$

$$=\frac{1}{2a}\int\frac{x^2+a}{x^4+a^2}dx-\frac{1}{2a}\int\frac{x^2-a}{x^4+a^2}dx=\frac{1}{2a}(I_1-I_2)$$

We have, $I_1=\frac{x^2+a}{x^4+a^2}dx=\int\frac{1+\frac{a}{x^2}}{x^2+\frac{a^2}{x^2}}dx$

$$=\int\frac{1+\frac{a}{x^2}}{\left(x-\frac{a}{x}\right)^2+2a}dx$$

Put $x-\frac{a}{x}=y\Rightarrow\left(1+\frac{a}{x^2}\right)dx=dy$

$\therefore$ $$I_1=\int\frac{dy}{y^2+2a}$$

$$=\frac{1}{\sqrt{2a}}\tan^{-1}\left(\frac{y}{\sqrt{2a}}\right)+C_1$$

$$=\frac{1}{\sqrt{2a}}\tan^{-1}\left(\frac{x^2-a}{x\sqrt{2a}}\right)+C_1$$

$\therefore$ $$I_2=\int\frac{x^2-a}{x^4+a^2}dx=\int\frac{1-\frac{a}{x^2}}{x^2+\frac{a^2}{x^2}}dx$$

$$=\int\frac{1-\frac{a}{x^2}}{\left(x+\frac{a}{x}\right)^2-2a}dx$$

Put $x+\frac{a}{x}=y\Rightarrow\left(1-\frac{a}{x^2}\right)dx=dy$

$\therefore$ $$I_2=\int\frac{dy}{y^2-2a}$$

$$=\frac{1}{2\sqrt{2a}}\log\left|\frac{y-\sqrt{2a}}{y+\sqrt{2a}}\right|+C_2$$

$$=\frac{1}{2\sqrt{2a}}\log\left|\frac{x^2-\sqrt{2a}x+a}{x^2+\sqrt{2a}x+a}\right|+C_2$$

$\therefore$ $$I=\frac{1}{2a\sqrt{2a}}\left[\tan^{-1}\left(\frac{x^2-a}{x\sqrt{2a}}\right)-\frac{1}{2}\log\left|\frac{x^2-\sqrt{2a}x+a}{x^2+\sqrt{2a}x+a}\right|\right]+C$$

where $C=(C_1-C_2)/2$ is another arbitrary constant.

Integration by Parts

This method is mainly used when the integral is the product of two functions. Let $f(x)$ and $g(x)$ be two differentiable functions of x. Then, we have

$$\int f(x)g(x)\,dx=f(x)\left[\int g(x)\,dx\right]-\int\left[\frac{d}{dx}f(x)\cdot\int g(x)\,dx\right]dx$$

= first function × (integral of the second function)

− integral of second function (integral of the first function × derivative of the second function)

While integrating by parts we use the order ILATE *i.e.,* inverse function, logarithmic function, algebraic function, trigonometric function, exponent function.

Integrals of Some Special Function

(i) $\int e^x\{f(x)+f'(x)\}\,dx=e^x f(x)+C$

(ii) $\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+C$

(iii) $\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)+C$

(iv) $\int\frac{f'(x)}{f(x)}dx=\log|f(x)|+C$

Example 12. Evaluate $\int x\log x\,dx$

(a) $\frac{x^2}{2}\log x + \frac{x^4}{4} + C$

(b) $\frac{x^2}{4}\log x - \frac{x^2}{4} + C$

(c) $\frac{x^2}{2}\log x - \frac{x^2}{4} + C$

(d) None of the above

Sol. (c) Taking $\log x$ as the first function and x as the second function.

Since,

$\therefore$ $$I = \int x\log x\,dx$$

$\Rightarrow$ $$I = \frac{x^2}{2}\log x - \int \frac{x^2}{2}\left(\frac{1}{x}\right)dx$$

$$= \frac{x^2}{2}\log x - \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}\log x - \frac{x^2}{4} + C$$

Example 13. Evaluate $\int \tan^{-1}x\,dx$

(a) $x\tan^{-1}x + \frac{1}{2}\log(1+x^2) + C$

(b) $x\tan^{-1}x - \frac{1}{2}\log(1+x^2) + C$

(c) $x\tan^{-1}x + \frac{1}{2}\log(1-x^2) + C$

(d) None of the above

Sol. (b) Taking $\tan^{-1}x$ as the first function and 1 as the second function.

$\therefore$ $$I = \int 1\cdot\tan^{-1}x\,dx$$

$$I = x\tan^{-1}x - \int\frac{x}{1+x^2}dx$$

Put $1+x^2 = y \Rightarrow x\,dx = \frac{dy}{2}$

$\therefore$ $$I = x\tan^{-1}x - \frac{1}{2}\int\frac{dy}{y}$$

$$= x\tan^{-1}x - \frac{1}{2}\log y + C$$

$$= x\tan^{-1}x - \frac{1}{2}\log(1+x^2) + C$$

Example 14. Evaluate $\int e^x(x+1)\,dx$

(a) $e^x + C$ (b) $xe^x + C$

(c) $e^x + x + C$ (d) $e^x + 1 + C$

Sol. (b) We know that, $\frac{d}{dx}\{x\} = 1$

So, using the formula $\int e^x\{f(x) + f'(x)\}\,dx = e^x f(x) + C$

Assuming $f(x) = x, f'(x) = 1$, we get $\int e^x(x+1)\,dx = e^x\cdot x + C$

Example 15. Evaluate $\int \frac{3x^2+8x}{x^3+4x^2+3}dx$

(a) $\frac{1}{3}(x^3+4x^2+3)^{1/3} + C$

(b) $\frac{1}{2}\log(x^3+4x^2+3) + C$

(c) $\log(x^3+4x^2+3) + C$

(d) $\frac{1}{2}\log|3x^2+8x| + C$

Sol. (c) We know that,

$$\frac{d}{dx}(x^3+4x^2+3) = 3x^2+8x$$

By using the formula $\int\frac{f'(x)}{f(x)}dx = \log|f(x)| + C$

We get, $$\int\frac{(3x^2+8x)dx}{x^3+4x^2+3} = \log|x^3+4x^2+3| + C$$

Integration of Rational Functions

If the integrand of is a rational function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

(or we can reduced to this form), where $P(x)$ and $Q(x)$ are polynomials in x then it is sometimes possible to express the integrals in terms of elementary functions by partial fraction etc.

Example 16. Evaluate the integral

$$\int\frac{\sin x\,dx}{(1+\cos x)(2+\cos x)}$$

(a) $\log\frac{(1+\cos x)}{(2+\cos x)} + C$ (b) $\log\frac{(2+\cos x)}{(1+\cos x)} + C$

(c) $\log\frac{(1+\sin x)}{(2+\sin x)} + C$ (d) $\log\frac{(2+\sin x)}{(1+\sin x)} + C$

Sol. (b) Put $\cos x = y \Rightarrow -\sin x\,dx = dy$

$\therefore$ $$I = -\int\frac{dy}{(1+y)(2+y)}$$

$\Rightarrow$ $$I = -\int\left[\frac{1}{1+y} - \frac{1}{2+y}\right]dy$$

$$= -[\log(1+y) - \log(2+y)] + C$$

$\Rightarrow$ $$I = \log\left(\frac{2+y}{1+y}\right) + C$$

$\Rightarrow$ $$I = \log\frac{(2+\cos x)}{(1+\cos x)} + C$$

Example 17. What is the value of $\int \frac{dx}{(x^2+a^2)(x^2+b^2)}$? *(NDA 2007 I)*

(a) $\frac{\frac{\tan^{-1}\left(\frac{x}{a}\right)}{a} - \frac{\tan^{-1}\left(\frac{x}{b}\right)}{b}}{(a^2+b^2)} + C$

(b) $\frac{\frac{\tan^{-1}\left(\frac{x}{a}\right)}{a} + \frac{\tan^{-1}\left(\frac{x}{b}\right)}{b}}{(a^2+b^2)} + C$

(c) $\frac{\frac{\tan^{-1}\left(\frac{x}{a}\right)}{a} - \frac{\tan^{-1}\left(\frac{x}{b}\right)}{b}}{(b^2-a^2)} + C$

(d) $\frac{\frac{\tan^{-1}\left(\frac{x}{a}\right)}{a} + \frac{\tan^{-1}\left(\frac{x}{b}\right)}{b}}{(b^2-a^2)} + C$

Sol. (c) $I = \int \frac{dx}{(x^2+a^2)(x^2+b^2)}$

$$= \frac{1}{(b^2-a^2)}\int \frac{b^2-a^2}{(x^2+a^2)(x^2+b^2)}dx$$

$$= \frac{1}{(b^2-a^2)}\int \left\{\frac{1}{x^2+a^2} - \frac{1}{x^2+b^2}\right\}dx$$

$$= \frac{1}{b^2-a^2}\left\{\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) - \frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)\right\} + C$$

Exercise

1. $\int \frac{\sqrt{(\tan x)}}{\sin x \cos x}dx$ is equal to
   (a) $2\sqrt{(\cot x)} + C$ (b) $\sqrt{(\cot x)} + C$
   (c) $\sqrt{(\tan x)} + C$ (d) $2\sqrt{(\tan x)} + C$

2. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$ is equal to
   (a) $2\cos\sqrt{x} + C$ (b) $\sqrt{\{(\cos x)/x\}} + C$
   (c) $\sin\sqrt{x} + C$ (d) $2\sin\sqrt{x} + C$

3. What is $\int \sec x° dx$ equal to? *(NDA 2009 I)*
   (a) $\log(\sec x° + \tan x°) + C$
   (b) $\frac{\pi \log \tan\left(\frac{\pi}{4} + \frac{\pi}{2}\right)}{180°} + C$
   (c) $\frac{180° \log \tan\left(\frac{\pi}{4} + \frac{\pi}{2}\right)}{\pi} + C$
   (d) $\frac{180° \log \tan\left(\frac{\pi}{4} + \frac{\pi x}{360°}\right)}{\pi} + C$

4. What is $\int \tan^2 x \sec^4 x\, dx$ equal to? *(NDA 2009 I)*
   (a) $\frac{\sec^5 x}{5} + \frac{\sec^3 x}{3} + C$ (b) $\frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + C$
   (c) $\frac{\tan^5 x}{5} + \frac{\sec^3 x}{3} + C$ (d) $\frac{\sec^5 x}{5} + \frac{\tan^3 x}{3} + C$

5. If $\int \frac{dx}{f(x)} = \log\{f(x)\}^2 + C$, then what is $f(x)$ equal to? *(NDA 2008 I)*
   (a) $2x + \alpha$ (b) $x + \alpha$
   (c) $\frac{x}{2} + \alpha$ (d) $x^2 + \alpha$

6. What is $\int \log(x+1)\, dx$ equal to? *(NDA 2008 I)*
   (a) $x\log(x+1) - x + C$ (b) $(x+1)\log(x+1) - x + C$
   (c) $\frac{1}{x+1} + C$ (d) $\frac{\log(x+1)}{x+1} + C$

7. What is $\int (e^x + 1)^{-1} dx$ equal to? *(NDA 2008 II)*
   (a) $\log(e^x + 1) + C$ (b) $\log(e^{-x} + 1) + C$
   (c) $-\log(e^{-x} + 1) + C$ (d) $-(e^x + 1) + C$

8. What is $\int \frac{d\theta}{\sin^2\theta + 2\cos^2\theta - 1}$ equal to? *(NDA 2008 I)*
   (a) $\tan\theta + C$ (b) $\cot\theta + C$
   (c) $\frac{1}{2}\tan\theta + C$ (d) $\frac{1}{2}\cot\theta + C$

9. What is $\int \frac{a + b\sin x}{\cos^2 x}dx$ equal to? *(NDA 2009 II)*
   (a) $a\sec x + b\tan x + C$ (b) $a\tan x + b\sec x + C$
   (c) $a\cot x + b\,\text{cosec}\, x + C$ (d) $a\,\text{cosec}\, x + b\cot x + C$
   Where a, b and c are constants

10. What is $\int \frac{\log x}{(1 + \log x)^2}dx$ equal to? *(NDA 2009 II)*
    (a) $\frac{1}{(1 + \log x)^3} + C$ (b) $\frac{1}{(1 + \log x)^2} + C$
    (c) $\frac{x}{(1 + \log x)} + C$ (d) $\frac{x}{(1 + \log x)^2} + C$
    where C is a constant